MTH 304: Metric Spaces and Topology

Homework V

(Due 23/02)

1. Reading assignment: Read carefully through the proofs of Theorem 20.5 (page 123) and Example 2 (page 131) from Munkres.
2. Prove the converse of the sequence lemma as stated in 1.9 (xiv) of the lesson plan.
3. Prove assertion 1.9 (vii) of the lesson plan.
4. For $p \geq 1$, and for points $x, y \in \mathbb{R}^n$, define
$$d_p(x,y) = \|x-y\|_p = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}.$$
Assuming that d_p defines a metric on $\mathbb{R}^n$, show that it induces the standard topology.
5. Determine whether the following functions $\mathbb{R} \to \mathbb{R}^\infty$ are continuous in the product, box and uniform topologies.
   (a) $f(t) = (t, 2t, 3t, \ldots)$.
   (b) $g(t) = (t, t, t, \ldots)$.
   (c) $h(t) = (t, t/2, t/3, \ldots)$.
6. Let (X, d) be a metric space. Define a map $d' : X \times X \to \mathbb{R}$ by
$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)},$$
for all $x, y \in \mathbb{R}$. [Hint: Consider $f(x) = x/(x+1)$, for $x > 0$, and use the mean value theorem.]
   (a) Show that (X, d') is a metric space.
   (b) Show that $\mathcal{T}_d = \mathcal{T}_{d'}$.
7. Let (X, d_X) and (Y, d_Y) be metric spaces. Then a map $f : X \to Y$ such that
$$d_X(x,y) = d_Y(f(x), f(y)), \text{ for all } x, y \in X$$
is called an *isometry*.
   (a) Show that if $f : X \to Y$ is an isometry, then f is an embedding.
   (b) Show that a surjective isometry is a homeomorphism.
8. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by
$$f_n(x) = \frac{1}{n^3(x - (1/n))^2 + 1}.$$
   (a) Show that f_n converges pointwise to the zero function.
   (b) Show that f_n does not converge uniformly to the zero function.